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Approximation of Continuous Functions by Polynomials with Integral Coefficients

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INTRODUCTION

Let $Q(Z)$ be the set of all polynomials with integral coefficients, let $-\infty < a < b < \infty$, let $C(a, b)$ denote the set of all real valued continuous functions defined on $[a, b]$, and let $f \in C(a, b)$ be arbitrary but fixed.

DEFINITION 1. (a) f is approximable on $[a, b]$ if and only if for each $\eta > 0$ there exists a $Q \in Q(Z)$ such that $|f(x) - Q(x)| < \eta$ for all $x \in [a, b]$.

(b) f is matchable on a set S if and only if $S \subseteq [a, b]$, and there exists a $Q \in Q(Z)$ such that $f(x) = Q(x)$ for all $x \in S$.

(c) For each $g \in C(a, b)$, define $\| g \| = \max_{a \leqslant x \leqslant b} | g(x)|$.

(d) Let $U(a, b) = \{Q \mid Q \in Q(Z), 0 \leqslant Q(x) < 1 \text{ for all } x \text{ in } [a, b], Q \not\equiv 0\}$. If $U(a, b) \neq \phi$, then let $J(a, b) = \{x \mid x \in [a, b], Q(x) = 0 \text{ for all } Q \in U(a, b)\}$ and call the points of $J(a, b)$ critical points of $[a, b]$.

The general question to be investigated can be stated as follows: does there exist a $Q_0 \in Q(Z)$ such that $\|f - Q_0\| \leqslant \|f - Q\|$ for all $Q \in Q(Z)$?; such a Q_0 would be called a best approximation to f on $[a, b]$. In this paper, results concerning (1) the existence of Q_0, (2) the uniqueness of Q_0, (3) the construction of Q_0, and (4) the magnitude of $\|f - Q_0\|$ will be developed.

The related topic of the existence of arbitrarily good approximations to f by elements of $Q(Z)$ has a rather long history. The problem was first raised by J. Pál [11] for the case in which $[a, b] = [-\alpha, \alpha]$, $|\alpha| < 1$. He has shown that $f(0) =$ an integer is a necessary and sufficient condition for the approximability of f on $[-\alpha, \alpha]$. S. Kakeya [8] studied the problem for the interval $[-1, 1]$, and Fekete and Lukács (in 1916, unpublished; see [3]) considered the problem for arbitrary intervals $[a, b]$. Y. Okada [10], S. N. Bernstein [2], L. Kantorovič [9], M. Fekete [3, 4, 5, and 6], I. Yamamoto [12], E. Hewitt and H. Zuckerman [7], and G. Andria [1] have also studied this problem and related ones.

Several known results will now be stated for later reference:

(a) Let $U(a, b) \neq \phi$; then $J(a, b)$ is finite. Furthermore, f is approximable on $[a, b]$ if and only if it is matchable on $J(a, b)$ ([7], Theorem 2.6).

(b) If $b - a \geqslant 4$, then f is approximable if and only if it coincides on $[a, b]$ with an element of $Q(Z)$ ([3], Theorem 1). If $b - a < 4$, then $U(a, b)$ is nonvoid (follows from [5], Theorem XIV), $J(a, b)$ is an n-point set ($n \geqslant 0$) and f is approximable if and only if the unique polynomial of degree $n - 1$ matching f on $J(a, b)$ has integral coefficients ([7], Theorem 4.3). (The unique polynomial of degree -1 is 0.)

(c) Let $J'(a, b)$ be the set of zeros of the polynomials $Q^* \in Q(Z)$ having leading coefficient one and all their zeros in $[a, b]$.

(d) If $b - a < 4$, then $J(a, b) = J'(a, b)$ ([7], 3.10).

(e) Let $-2 \leqslant a < b \leqslant 2$. Then $J'(a, b) = (\{-2, 2\} \cap [a, b]) \cup (\bigcup' T_k)$, where $\bigcup'$ is the union over all k such that $k \geqslant 3$, $x_{1k} \leqslant b$, $x_{\ell k} \geqslant a$ and

$$\ell = \begin{cases} \dfrac{k-1}{2} & \text{if } k \text{ is odd,} \\ \dfrac{k-2}{2} & \text{if } k \equiv 0 \pmod 4, \\ \dfrac{k-4}{2} & \text{if } k \equiv 2 \pmod 4, \end{cases} \qquad T_k = \{2\cos(2\pi j/k) \mid 0 \leqslant j \leqslant k/2, \ (j, k) = 1\},$$

$x_{ij} = 2\cos(2\pi i/j)$. ([7], 5.5).

(f) Let $\gamma = [a] + 2$. Then if $b - \gamma \leqslant 2$, we obtain $J'(a, b)$ upon translating $J'(a - \gamma, b - \gamma)$ by γ; that is, $J'(a, b) = J'(a - \gamma, b - \gamma) + \gamma$. Since $J'(a - \gamma, b - \gamma)$ is identified in (e) above, $J'(a, b)$ is identified in this case [7, 5.7].

Approximable Functions and Best Approximations

A fundamental relationship between approximable functions and best approximations is clearly demonstrated in the following theorem:

Theorem 1. *If f is approximable on $[a, b]$, then either $f \in Q(Z)$ or there does not exist a best approximation to f on $[a, b]$.*

Proof. Suppose f is not in $Q(Z)$ but is approximable on $[a, b]$. Assume there exists a best approximation, Q_0, to f on $[a, b]$. Choose $x_0 \in [a, b]$ such that $f(x_0) \neq Q_0(x_0)$, and let $\eta = \frac{1}{2} | f(x_0) - Q_0(x_0)| > 0$. Then for any $Q \in Q(Z)$, $\| f - Q \| \geqslant \| f - Q_0 \| \geqslant | f(x_0) - Q_0(x_0)| > \eta > 0$. This, however, contradicts the approximability of f on $[a, b]$.

The construction of an approximable function from an arbitrary given continuous function is now considered.

THEOREM 2. *Let $b - a < 4$ and let $L(x)$ be the Lagrange interpolation polynomial to f in $J(a, b)$. (For $J(a, b) = \phi$, set $L(x) \equiv 0$.) Then $g(x) \equiv f(x) - L(x)$ is approximable on $[a, b]$.*

Proof. $g(x) = 0$ for all $x \in J(a, b)$. Thus, $g(x)$ is matchable on $J(a, b)$ and, hence, approximable on $[a, b]$.

If L of Theorem 2 is replaced by any function $\ell \in C(a, b)$ such that $\ell(x) = f(x)$ for all $x \in J(a, b)$, then $f - \ell$ is also approximable on $[a, b]$.

THEOREM 3. *Let $b - a < 4$. Then f is approximable on $[a, b]$ if and only if L is approximable on $[a, b]$.*

This follows at once from Theorem 2.

An immediate consequence of result (b) above is that if $b - a < 4$, then $f(x)$ is approximable if and only if $L(x)$ has integral coefficients. Thus, if $b - a < 4$ and if there exists a sequence $\{Q_i\}$ such that $Q_i \in Q(Z)$ and $\| Q_i - L \| \to 0$ as $i \to \infty$, then $L \in Q(Z)$.

ON BEST APPROXIMATIONS

THEOREM 4. *A best approximation to a continuous function is, in general, not unique.*

Proof. Let $[a, b] = [0, 1]$, $f(x) = -x + \frac{1}{2}$, $Q_1(x) = 0$, and $Q_2(x) = 2f(x)$. Then $\| f - Q_1 \| = \| f - Q_2 \| = \frac{1}{2}$. Now let $Q(x) = \sum_{i=0}^{n} a_i x^i$ be any polynomial with integral coefficients. Then $\| f - Q \| \geqslant | f(0) - \sum_{i=0}^{n} a_i 0^i | = | \frac{1}{2} - a_0 | \geqslant \frac{1}{2}$, since a_0 is an integer. Hence, both Q_1 and Q_2 are best approximations to f on $[0, 1]$.

The question of existence of best approximations cannot be answered as easily as that of uniqueness. However, some insight into the existence problem can be achieved by studying the question: If Q is an approximation to f, what general procedures could be used to hopefully improve upon Q? The following theorem and its proof contain an indication in this direction.

THEOREM 5. *Suppose $b - a < 4$, $J(a, b) \neq \phi$; let Q be an arbitrary element of $Q(z)$, and set $\mu = \max_{J(a,b)} | f(x) - Q(x)|$. Then, for each $\eta > 0$, there exists a polynomial P_η in $Q(Z)$ such that $\| f - P_\eta \| - \eta \leqslant \mu$.*

Proof. Let $\eta > 0$ be arbitrary but fixed. Define a function $G(x)$ as follows: (1) $G(x) = f(x) - Q(x)$ throughout $E = \{x \mid |f(x) - Q(x)| \geqslant \mu + (\eta/2)\}$, (2) $G(x) = 0$ throughout $E_0 = J(a, b) \cup \{x \mid f(x) - Q(x) = 0\}$, and (3) $G(x)$ is linear on each of the disjoint intervals whose union is $E^* = [a, b] - E_0 - E$. By a proper definition in (3), G will be continuous in $[a, b]$. Also, if x belongs to $[a, b] - E$, then $f(x) - Q(x)$ and $G(x)$ have the same sign; hence $|f(x) - Q(x) - G(x)| \leqslant \max\{|f(x) - Q(x)|, |G(x)|\} \leqslant \mu + (\eta/2)$. Also, G is matchable on $J(a, b)$ and thus approximable on $[a, b]$; hence, there exists Q_η in $Q(Z)$ such that $\| G(x) - Q_\eta(x)\| < \eta/2$. Therefore,

$$\begin{aligned}\| f(x) - (Q_\eta(x) + Q(x))\| &\leqslant \| f(x) - Q(x) - G(x)\| + \| G(x) - Q_\eta(x)\| \\ &< \mu + (\eta/2) + \eta/2 = \mu + \eta.\end{aligned}$$

Theorem 5 shows that if $\mu < \| f - Q \|$, there exists a better approximation to f in $Q(Z)$. This theorem will also be used to prove the following:

THEOREM 6. *Let $b - a < 4$, $f \notin Q(Z)$, and let Q_0 be a best approximation to f. Then*: (1) *f is not approximable on* $[a, b]$; (2) $\max_{J(a,b)} |f(x) - Q_0(x)| = \| f - Q_0 \|$.

Proof. (1) follows from Theorem 1. Also, by Theorem 2, $J(a, b) \neq \phi$. By Theorem 5, for each $\eta > 0$ there exists some P_η in $Q(Z)$ such that $\| f - P_\eta \| - \eta \leqslant \max_{J(a,b)} |f(x) - Q_0(x)|$. Hence, $\| f - Q_0 \| \leqslant \| f - P_\eta \| \leqslant \max_{J(a,b)} |f(x) - Q_0(x)| + \eta$. Since $\eta > 0$ is arbitrary,

$$\| f - Q_0 \| = \max_{J(a,b)} |f(x) - Q_0(x)|.$$

Theorem 6 identifies some of the points of maximum deviation of a best approximation: a best approximation assumes its maximum deviation from f on some subset of the set of critical points of the interval. In addition, this theorem can sometimes be used to determine that a particular approximation is not best.

THEOREM 7. *Let $b - a < 4$ and suppose f is not approximable on* $[a, b]$. *Then there exists an $\eta_0 > 0$ such that for every Q in $Q(Z)$, $|f(x) - Q(x)| > \eta_0$ for some x in $J(a, b)$.*

Proof. Since f is not approximable on $[a, b]$, there exists an $\eta' > 0$ such that for each Q in $Q(Z)$, $|f(x) - Q(x)| > \eta'$ for some x in $[a, b]$. Assume that for each $\eta > 0$ there exists Q_η in $Q(Z)$ such that $|f(x) - Q_\eta(x)| \leqslant \eta$ for all x in $J(a, b)$. Take $\eta = \eta'/2$. By Theorem 5, there exists P_η in $Q(Z)$ such that $\| f - P_\eta \| - \eta \leqslant \max_{J(a,b)} |f(x) - Q_\eta| \leqslant \eta$. Thus $\| f - P_\eta \| \leqslant 2\eta = \eta'$; but this contradicts the first part of the proof.

THEOREM 8. *There exists an* $[a, b]$ *with* $b - a < 4$, *and an* f *belonging to* $C(a, b)$ *such that* f *is not approximable on* $[a, b]$, *and such that there does not exist a best approximation to* f *on* $[a, b]$.

Proof. Define functions f, g on $[-\frac{1}{2}, \frac{1}{2}]$ by

$$g(x) = \begin{cases} \dfrac{1}{1+x} & \text{for} \quad x \in [0, \frac{1}{2}], \\ \dfrac{1}{1-x} & \text{for} \quad x \in [-\frac{1}{2}, 0], \end{cases}$$

and $f(x) = g(x) - (\frac{1}{4})$. Since $J(-\frac{1}{2}, \frac{1}{2}) = \{0\}$ [7, p. 317, 5.5 Theorem] and $g(0) = 1$, $g(x)$ is approximable on $[-\frac{1}{2}, \frac{1}{2}]$. By Theorem 3, f is not approximable there. Suppose there exists a best approximation Q_0 to f on $[-\frac{1}{2}, \frac{1}{2}]$. Since g is approximable, for each $\eta > 0$ there exists Q_η in $Q(Z)$ such that $|g(x) - Q_\eta(x)| = |f(x) + \frac{1}{4} - Q_\eta(x)| < \eta$ throughout $[-\frac{1}{2}, \frac{1}{2}]$. Hence, for every $\eta > 0$, $\|f - Q_0\| \leqslant \|f - Q_\eta\| < \eta + (\frac{1}{4})$, and so, $\|f - Q_0\| \leqslant \frac{1}{4}$. Since $f(0) = \frac{3}{4}$, we must have $Q_0(0) = 1$; consequently, $\|f - Q_0\| = \frac{1}{4}$.

There exists a nondegenerate interval $[-d, d] \subseteq [-\frac{1}{2}, \frac{1}{2}]$ such that in each of $[-d, 0]$, $[0, d]$ exactly one of the following holds: (1) Q_0 is constant, (2) Q_0 is strictly increasing, (3) Q_0 is strictly decreasing. However, since f is strictly decreasing on $[0, \frac{1}{2}]$, Q is strictly decreasing on $[0, d]$ since in all other cases we would have, throughout $[0, d]$,

$$|f(x) - Q_0(x)| > |f(0) - Q_0(0)| = \tfrac{1}{4},$$

which would contradict the equality $\|f - Q_0\| = \frac{1}{4}$. Similarly, Q_0 is strictly increasing on $[-d, 0]$. Hence, Q_0 has a relative maximum at $x = 0$. There exists an h, $0 < h \leqslant d$, such that throughout $(0, h)$ we have $-\frac{1}{2} \leqslant Q_0'(x) \leqslant 0$ and $-1 \leqslant f'(x) = -1/(1 + x)^2 \leqslant -\frac{4}{9}$. Thus, throughout $(0, h)$,

$$(Q_0(x) - f(x))' > 0,$$

and therefore $\|f - Q_0\| \geqslant Q_0(h) - f(h) > Q_0(0) - f(0) = \frac{1}{4}$, contradicting the equality $\|f - Q_0\| = \frac{1}{4}$.

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